Caringbah High School

Year 12 2022
Mathematics Extension 2
HSC Course
Assessment Task 4 – Trial HSC Examination

General Instructions
- Reading time – 10 minutes
- Working time – 3 hours
- Write using black or blue pen
- NESA-approved calculators may be used
- A reference sheet is provided
- In Questions 11-16, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 100

Section I **10 marks**
Attempt Questions 1-10
Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **90 marks**
Attempt Questions 11-16
Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: ______________________________ **Class:** __________

Marker's Use Only							
Section I	**Section II**						**Total**
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**	**Q15**	**Q16**	
/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet (page 13) for Questions 1–10

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1. Let the point P on an Argand diagram represent the complex number z.
After being multiplied by another complex number, ω, P is rotated 90° anti-clockwise and z is enlarged by a factor of 3.
Which of the following is the value of ω?

(A) $3i$

(B) $-3i$

(C) e^{-3i}

(D) $3e^{-\frac{\pi}{2}i}$

2. A particle moves in a straight line so that its displacement, x metres, at any time, t seconds, is given by $x = 5\sin 3t + 12\cos 3t$.
What is the speed of the particle as it passes through the centre of its motion?

(A) 12 m/s

(B) 13 m/s

(C) 39 m/s

(D) 117 m/s

3. Which of the following is equivalent to the expression $\dfrac{12x-3}{(x-2)(x^2-3x+2)}$?

(A) $\dfrac{9}{x-1} + \dfrac{9}{x-2} - \dfrac{21}{(x-2)^2}$

(B) $\dfrac{9}{x-1} + \dfrac{18}{x-2} - \dfrac{21}{(x-2)^2}$

(C) $\dfrac{9}{x-1} - \dfrac{9}{x-2} + \dfrac{21}{(x-2)^2}$

(D) $\dfrac{9}{x-1} - \dfrac{18}{x-2} + \dfrac{21}{(x-2)^2}$

4. Which of the following is the vector equation of a line that passes through the point $(1,3,-2)$ and is perpendicular to the line $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$?

(A) $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

(B) $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

(C) $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

(D) $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$

5. The acceleration of a particle is given by $a = e^{-3t}$ m/s^2 where the particle has an initial velocity of 2 m/s. What is the terminal velocity of the particle?

(A) $1.\dot{3}$ m/s

(B) $1.\dot{6}$ m/s

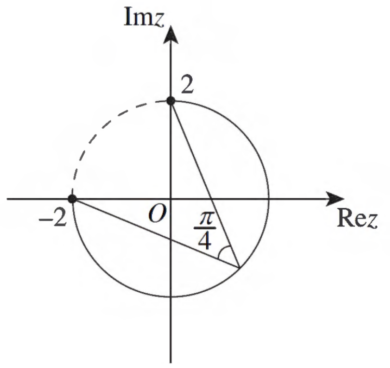
(C) $2.\dot{3}$ m/s

(D) $2.\dot{6}$ m/s

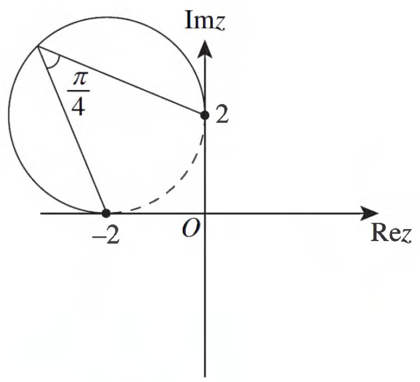
Section I continues on page 5

6. Which of the following diagrams best represents the solutions to the equation $\arg\left(\dfrac{z-2}{z+2i}\right)=\dfrac{\pi}{4}$?

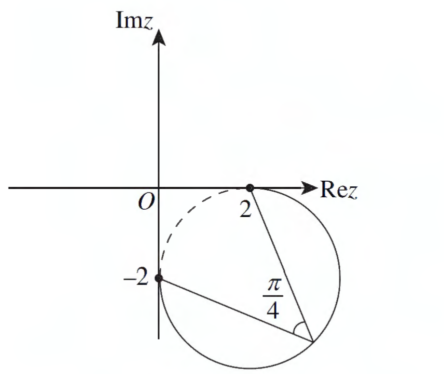
(A)



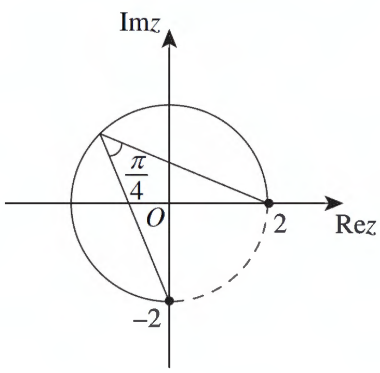
(B)



(C)



(D)



7. The points A, B and C are collinear where $\overrightarrow{OA} = \underset{\sim}{i} - \underset{\sim}{j}$, $\overrightarrow{OB} = -3\underset{\sim}{j} - \underset{\sim}{k}$ and $\overrightarrow{OC} = 2\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}$ for some constants a and b. What are the values of a and b?

(A) $a = -1$ and $b = -1$

(B) $a = -1$ and $b = 1$

(C) $a = 1$ and $b = -1$

(D) $a = 1$ and $b = 1$

8. Consider the statement: '$\exists x \in \mathbb{R},\ \ln x = 1$ and $x > 2$'
Which of the following is the negation of the statement?

(A) $\exists x \in \mathbb{R},\ \ln x \neq 1$ or $x \leq 2$

(B) $\exists x \in \mathbb{R},\ \ln x \neq 1$ and $x \leq 2$

(C) $\forall x \in \mathbb{R},\ \ln x \neq 1$ or $x \leq 2$

(D) $\forall x \in \mathbb{R},\ \ln x \neq 1$ and $x \leq 2$

9. If $\int_1^4 f(x)\,dx = k$ for some constant k, what is the value of $\int_1^4 f(5-x)\,dx$?

(A) $-k$

(B) $5-k$

(C) $k+5$

(D) k

10. Which of the following best describes the path of a particle with the following parametric equations?

$$\begin{cases} x = t\sin t \\ y = t\cos t \\ z = t \end{cases}$$

(A) spiral around the z-axis, traversing in a clockwise direction

(B) spiral around the z-axis, traversing in an anticlockwise direction

(C) helix around the z-axis, traversing in a clockwise direction

(D) helix around the z-axis, traversing in an anticlockwise direction

End of Section I

Section II

60 marks
Attempt Questions 11–16
Allow about 2 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the complex numbers $z = 2 + 2\sqrt{3}i$ and $w = 2\sqrt{3} - 2i$.

(i) Find z and w in modulus-argument form. **2**

(ii) Find zw and $\dfrac{z}{w}$ in modulus-argument form. **2**

(iii) Describe the relationship between z and w geometrically. **1**

(b) Consider the equation $f(z) = 0$ where $f(z) = z^3 - 11z^2 + 55z - 125$.

(i) Find the three roots of the equation in the form $a + ib$, where a and b are real. **3**

(ii) Show that the points A, B and C in the Argand diagram representing these roots lie on a circle of the form $|z| = k$ for some constant k, and find the area of ΔABC. **2**

(c) **(i)** Find constants A and B such that **2**

$$A(3\sin x + 2\cos x) + B(3\cos x - 2\sin x) = 8\sin x + 14\cos x$$

(ii) Hence, find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{8\sin x + 14\cos x}{3\sin x + 2\cos x}\,dx$ **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) We are given that $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ are three consecutive terms in a geometric series, where $a,\ b,\ c \in \mathbb{R}$. Show that $a^2 + c^2 \geq ab + bc$. **3**

(b) A body of unit mass falls under gravity through a resisting medium. The body falls from rest from a height above the ground. The resistance to its motion is $\frac{1}{100}v^2$ where v m/s is the speed of the body when it has fallen a distance x m. The acceleration due to gravity is g m/s^2.

(i) Show that the equation of motion of the body is $\ddot{x} = g - \frac{1}{100}v^2$ **1**

(ii) Show that the terminal velocity V m/s of the body is given by $V = 10\sqrt{g}$ **1**

(iii) Hence show that $v^2 = V^2\left(1 - e^{-\frac{x}{50}}\right)$ **3**

(iv) Find the distance fallen in metres until the body reaches a velocity equal to half of its terminal velocity. **2**
(You may assume this occurs before the body reaches the ground.)

(c) **(i)** Use the substitution $u = \frac{1}{x}$ to show that $\int_{\frac{1}{a}}^{a} \frac{\ln x}{1+x^2}\, dx = 0$ **2**

for any constant $a > 0$.

(ii) Hence, find $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\tan^{-1} x}{x}\, dx$ in simplest exact form. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the lines L_1 and L_2, determined by the vector equations

$$L_1:\ \underset{\sim}{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad L_2:\ \underset{\sim}{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Show that L_1 and L_2 intersect and are perpendicular, stating the coordinates of the point of intersection. **3**

(b) A body of mass m kg is travelling in a horizontal straight line so that the resultant force on the body is a resistance force of magnitude $\frac{1}{10} m\sqrt{1+v}$ when its speed is v m/s. Initially, the speed of the body is 15 m/s.

(i) Find the time taken for the body to come to rest. **2**

(ii) Find the distance travelled by the body in coming to rest. **3**

(c) **(i)** Find the real numbers A and B such that

$$\frac{5}{(x+3)(2x+1)} = \frac{A}{x+3} + \frac{B}{2x+1}$$ **2**

(ii) Hence, or otherwise, evaluate

$$\int_0^2 \frac{5}{(x+3)(2x+1)}\, dx$$ **2**

(d) Find the points of intersection between the sphere S and the line L, given below. **3**

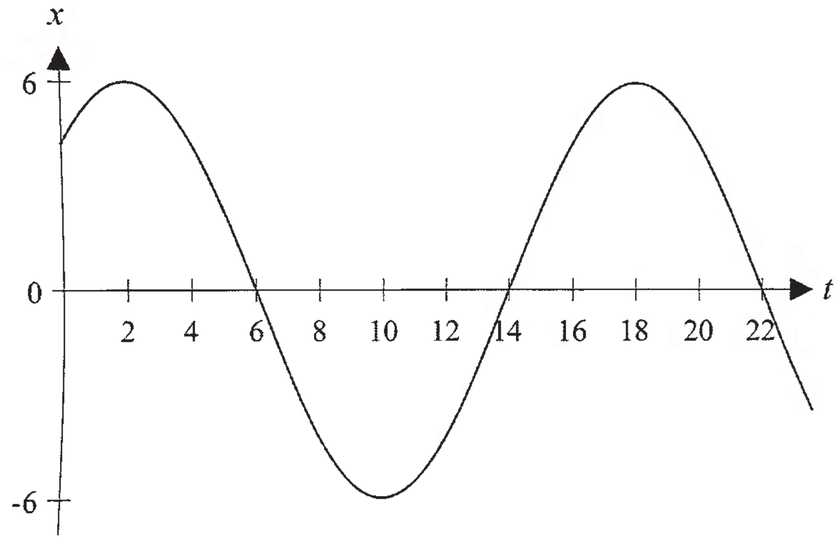
$$S: \quad \left| \underset{\sim}{r} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right| = 3$$

$$L: \quad \underset{\sim}{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The graph below shows the displacement x cm from the centre of motion at time t seconds for a particle performing simple harmonic motion in a straight line.



(i) Express displacement as a function of time in the form $x = A\sin\left(nt+\alpha\right)$ **2**

(ii) Find the distance travelled by the particle in the first 30 seconds of its motion after observation began at time $t=0$. **2**

(b) **(i)** If $t = \tan\theta$, prove that $\tan 4\theta = \dfrac{4t\left(1-t^2\right)}{1-6t^2+t^4}$ **2**

(ii) If $\tan\theta\tan 4\theta = 1$, deduce that $5t^4 - 10t^2 + 1 = 0$ **2**

(iii) Given that $\theta = \dfrac{\pi}{10}$ and $\theta = \dfrac{3\pi}{10}$ are roots of the equation $\tan\theta\tan 4\theta = 1$, find the exact value of $\tan\dfrac{\pi}{10}$. **2**

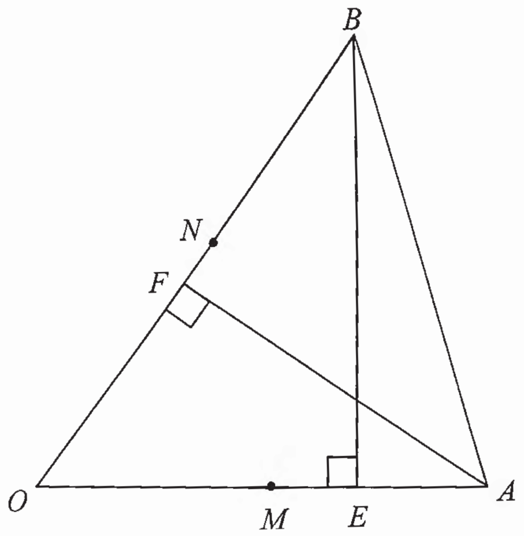
(c) **(i)** Show that for $k \geq 2$, $\tan^{-1}\left(\dfrac{1}{k-1}\right) - \tan^{-1}\left(\dfrac{1}{k+1}\right) = \tan^{-1}\left(\dfrac{2}{k^2}\right)$ **2**

(ii) Hence, show that $\displaystyle\lim_{n\to\infty}\left[\sum_{k=1}^{n}\tan^{-1}\left(\frac{2}{k^2}\right)\right] = \frac{3\pi}{4}$ **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) In ΔOAB, BE is the altitude from B to OA and AF is the altitude from A to OB. M and N are the midpoints of OA and OB respectively. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



Not to scale

Use vector methods to show that $|OM||OE| = |ON||OF|$ **3**

(b) Let $I_n = \int_0^1 \frac{x^n}{x^2+1}\,dx$ for $n \geq 2$

(i) Show that $I_n + I_{n-2} = \frac{1}{n-1}$ **2**

(ii) Hence, or otherwise, show that $\int_0^1 \frac{x^2(x-1)^2}{x^2+1}\,dx = \ln 2 - \frac{2}{3}$ **3**

(c) Consider the functions $f_n(x) = e^x - \left(1 + \sum_{r=1}^{n} \frac{x^r}{r!}\right)$, $n = 1, 2, 3, \ldots$

(i) Show that $f_n(0) = 0$ and $f_{n+1}{}'(x) = f_n(x)$ for $n = 1, 2, 3, \ldots$ **2**

(ii) Show that $f_1(x) > 0$ for all $x > 0$ and hence $1 + x < e^x$ for all $x > 0$ **2**

(iii) Use mathematical induction to show that for all positive integers $n \geq 1$,

$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} < e^x$ for all $x > 0$ **3**

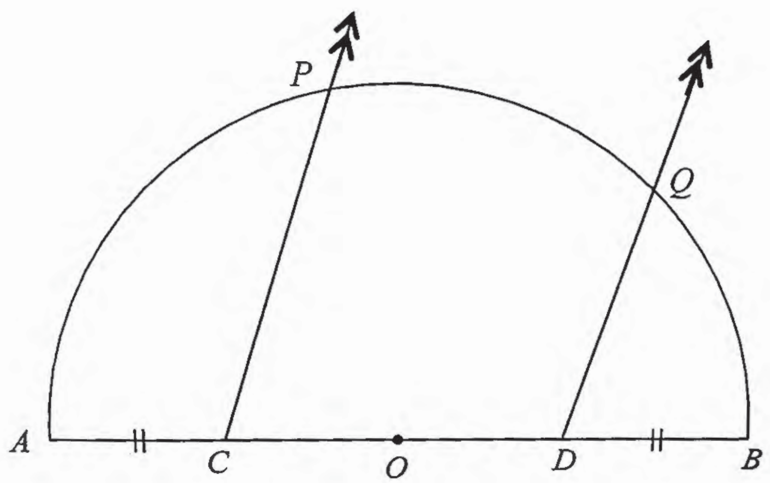
End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) **(i)** Show that $(1+i\tan\theta)^n+(1-i\tan\theta)^n=\dfrac{2\cos n\theta}{\cos^n\theta}$, where $\cos\theta\neq 0,\ n\in\mathbb{Z}^+$ **2**

(ii) Hence, show that the roots of $(1+z)^2+(1-z)^2=0$ are $z=\pm i\tan\dfrac{\pi}{4}$ when z is purely imaginary. **2**

(b) A semi-circle is drawn on diameter AB. O is the midpoint of AB and the points C and D lie on AB such that $AC=BD$. Parallel lines are drawn through C and D, intersecting the semi-circle at P and Q respectively. $\overrightarrow{OC}=\underset{\sim}{c}$ and $\overrightarrow{CP}=\underset{\sim}{p}$.



(i) Explain why $\overrightarrow{DQ}=\lambda\underset{\sim}{p}$ for some scalar $\lambda>0$, then show that $(1-\lambda)\,\underset{\sim}{p}\cdot\underset{\sim}{p}+2\,\underset{\sim}{c}\cdot\underset{\sim}{p}=0$ **3**

(ii) Hence show that $\angle CPQ=90°$. **2**

(c) **(i)** For integer values of n, $I_n=\displaystyle\int_1^e(1-\ln x)^n\,dx$ for $n\geq 0$.

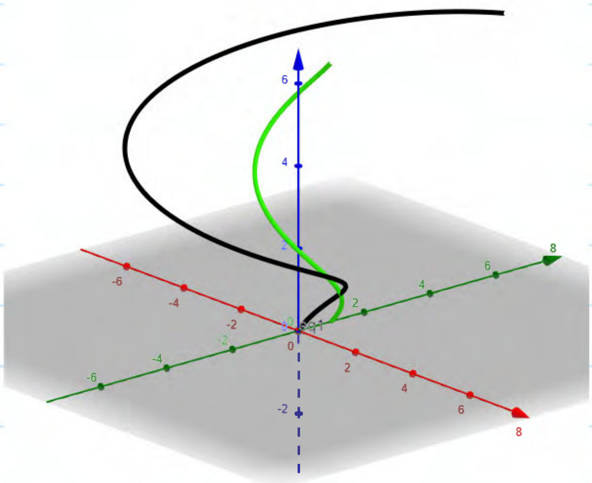
Show that $I_n=-1+nI_{n-1}$ for $n\geq 1$. **2**

(ii) Use mathematical induction to show that

$I_n=n!e-1-\displaystyle\sum_{r=1}^{n}{}^nP_r$ for all integers $n\geq 1$ **4**

End of Examination

Mathematics Extension 2 - Trial HSC 2022 Solutions



Section 1

1. A	2. C	3. C	4. C	5. C
6. D	7. D	8. C	9. D	10. A

Section 2

Question 11

(a) $z = 2+2\sqrt{3}i \qquad w = 2\sqrt{3}-2i$

(i) $|z| = \sqrt{2^2+(2\sqrt{3})^2} = 4$

$\arg(z) = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$ $\qquad \therefore z = 4\operatorname{cis}\frac{\pi}{3}$ ✓

$|w| = \sqrt{(2\sqrt{3})^2+(-2)^2} = 4$

$\arg(w) = \tan^{-1}\left(\frac{-2}{2\sqrt{3}}\right) = -\frac{\pi}{6}$ $\qquad \therefore w = 4\operatorname{cis}\left(-\frac{\pi}{6}\right)$ ✓

(ii) $zw = 4\operatorname{cis}\frac{\pi}{3} \times 4\operatorname{cis}\left(-\frac{\pi}{6}\right)$

$= 16\operatorname{cis}\frac{\pi}{6}$ ✓

$\frac{z}{w} = \frac{4\operatorname{cis}\pi/3}{4\operatorname{cis}(-\pi/6)}$

$= \operatorname{cis}\frac{\pi}{2}$ ✓

(iii) $\frac{z}{w} = \operatorname{cis}\frac{\pi}{2} = i \Rightarrow z = iw$ OR $w = -iz$

$\therefore$ z is w rotated anti-clockwise 90° ✓
OR w is z rotated clockwise 90°

(b) $f(z) = z^3 - 11z^2 + 55z - 125 = 0$

(i) Test $f(5) = 5^3 - 11(5)^2 + 55(5) - 125 = 0$

$\therefore z = 5$ is a root $\Rightarrow$ $(z-5)$ is a factor ✓

$$
\begin{array}{r}
z^2 - 6z + 25 \\
z-5 \overline{\big) z^3 - 11z^2 + 55z - 125} \\
\underline{z^3 - 5z^2} \qquad\qquad\quad \\
-6z^2 + 55z \qquad\quad \\
\underline{-6z^2 + 30z} \qquad\quad \\
25z - 125 \\
\underline{25z - 125} \\
0
\end{array}
$$

$\therefore f(z) = (z-5)\underbrace{(z^2 - 6z + 25)}$ ✓

$$z = \frac{6 \pm \sqrt{-64}}{2(1)} = \frac{6 \pm 8i}{2} = 3 \pm 4i$$

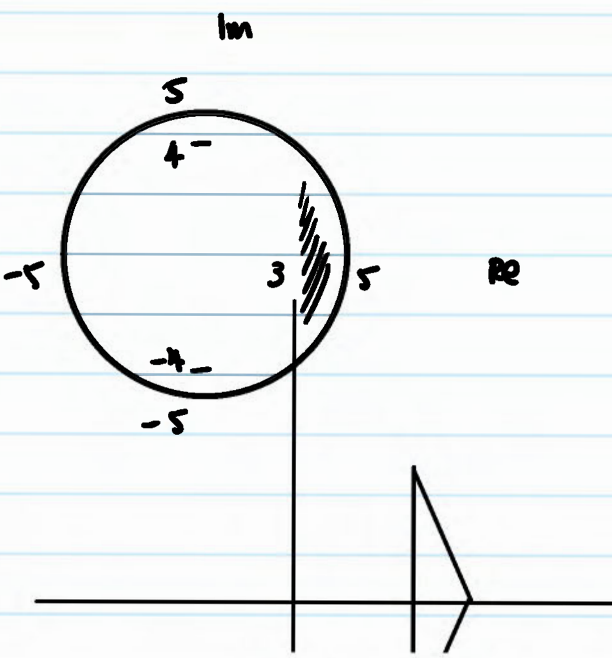
$\therefore z = 5,\ 3 \pm 4i$ ✓

(ii) $|5| = 5$

$|3+4i| = \sqrt{3^2 + 4^2} = 5$

$|3-4i| = \sqrt{3^2 + (-4)^2} = 5$

$\therefore$ Roots lie on circle $|z| = 5$ ✓



Area $= \frac{1}{2}bh$

$= \frac{1}{2} \times 2 \times 8$

$= 8$ units^2 ✓

(c)(i) $A(3\sin x+2\cos x)+B(3\cos x-2\sin x)=8\sin x+14\cos x$

$3A\sin x+2A\cos x+3B\cos x-2B\sin x=8\sin x+14\cos x$

$(3A-2B)\sin x+(2A+3B)\cos x=8\sin x+14\cos x$

Equating coefficients,

$3A-2B=8$ — ①

$2A+3B=14$ — ②

① × 2: $6A-4B=16$ — ③

② × 3: $6A+9B=42$ — ④

④ − ③: $13B=26$

$B=2$ — ⑤ ✓

Sub ⑤ in ①: $3A-2(2)=8$

$3A=12$

$A=4$ ✓

$\therefore A=4,\ B=2$

(ii) $$\int_0^{\pi/2}\frac{8\sin x+14\cos x}{3\sin x+2\cos x}\,dx$$

$$=\int_0^{\pi/2}\frac{4(3\sin x+2\cos x)+2(3\cos x-2\sin x)}{3\sin x+2\cos x}\,dx$$

$$=\int_0^{\pi/2}4+\frac{2(3\sin x+2\cos x)'}{3\sin x+2\cos x}\,dx$$ ✓

$$=\Big[4x+2\ln|3\sin x+2\cos x|\Big]_0^{\pi/2}$$ ✓

$$=\frac{4\pi}{2}+2\ln|3(1)+2(0)|-(0+2\ln|3(0)+2(1)|)$$

$$=2\pi+2\ln(3/2)$$ ✓

Question 12

(a) GP: $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ $\quad a,b,c \in \mathbb{R}$

Common ratio $r = \dfrac{\frac{1}{b}}{\frac{1}{a}} = \dfrac{\frac{1}{c}}{\frac{1}{b}}$

$$\frac{a}{b} = \frac{b}{c}$$

$$\therefore ac = b^2 \quad \text{—} (1)$$

✓

Now, consider $(a-b)^2 = a^2 - 2ab + b^2 \geq 0$

$$a^2 + b^2 \geq 2ab \quad \text{—} (2)$$

Similarly, $b^2 + c^2 \geq 2bc$ — (3)

$$a^2 + c^2 \geq 2ac \quad \text{—} (4)$$

✓

$(2)+(3)+(4)$: $2(a^2 + b^2 + c^2) \geq 2(ab + bc + ac)$

$a^2 + ac + c^2 \geq ab + bc + ac$ from (1)

$\therefore a^2 + c^2 \geq ab + bc$, as required ✓

(b)(i) $F = ma$

$g - \frac{1}{100}v^2 = 1 \times \ddot{x}$

$\therefore \ddot{x} = g - \frac{1}{100}v^2$

$\downarrow g \quad \uparrow \frac{1}{100}v^2$

✓ Show

(ii) When $\ddot{x} = 0$, $v = V > 0$

$0 = g - \frac{1}{100}V^2$

$\frac{1}{100}V^2 = g$

$V^2 = 100g$

$\therefore V = 10\sqrt{g}$

✓ Show

(iii) $\ddot{x} = v\frac{dv}{dx} = g - \frac{1}{100}v^2$

$$\frac{dv}{dx} = \frac{100g - v^2}{100v}$$

$$\frac{dx}{dv} = \frac{100v}{100g - v^2}$$

$$x = \int \frac{100v}{100g - v^2}\,dv$$ ✓

$$= -50\ln|100g - v^2| + C$$

At $t=0$, $x=0$, $v=0$

$$0 = -50\ln|100g - 0^2| + C$$

$$C = 50\ln 100g$$

$$\therefore x = 50\ln\left|\frac{100g}{100g - v^2}\right|$$ ✓

$$\frac{x}{50} = \ln\left|\frac{100g}{100g - v^2}\right|$$

$$e^{\frac{x}{50}} = \frac{100g}{100g - v^2}$$

$$100g - v^2 = 100g\,e^{-x/50}$$

$$v^2 = 100g(1 - e^{-x/50})$$

$$\therefore v^2 = V^2(1 - e^{-x/50})$$ ✓ Show

(iv) Sub $v = V/2$

$$\frac{\cancel{V^2}}{4} = \cancel{V^2}(1 - e^{-x/50})$$

$$e^{-x/50} = 3/4$$ ✓

$$-\frac{x}{50} = \ln\frac{3}{4}$$

$$x = 50\ln 4/3$$

$$\therefore x = 14.38 \text{ m (2dp)}$$ ✓

(c) $\displaystyle\int_{1/a}^{a} \frac{\ln x}{1+x^2}\,dx$

Let $u = 1/x \rightarrow x = 1/u$

$du = -1/x^2\,dx$

$dx = -x^2\,du$

$dx = -1/u^2\,du$

$\begin{cases} x = 1/a, & u = a \\ x = a, & u = 1/a \end{cases}$

$$= \int_{a}^{1/a} \frac{\ln(1/u)}{1+(1/u)^2} \cdot -\frac{1}{u^2}\,du$$

$$= \int_{1/a}^{a} \frac{-\ln u}{\frac{u^2+1}{\cancel{u^2}}} \cdot \frac{1}{\cancel{u^2}}\,du \quad ✓$$

$$= \int_{1/a}^{a} \frac{-\ln u}{1+u^2}\,du$$

$$= \int_{1/a}^{a} \frac{-\ln x}{1+x^2}\,dx \quad \text{Dummy variable}$$

$$2\int_{1/a}^{a} \frac{\ln x}{1+x^2}\,dx = 0 \quad ✓$$

$$\therefore \int_{1/a}^{a} \frac{\ln x}{1+x^2}\,dx = 0$$

(ii) $\displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{\tan^{-1}x}{x}\,dx$

$u = \tan^{-1}x$ $\quad v' = 1/x$

$u' = \frac{1}{1+x^2}$ $\quad v = \ln x$

$$= \left[\ln x \cdot \tan^{-1}x\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} - \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx \quad ✓$$

$$= \ln\sqrt{3}\cdot\frac{\pi}{3} - \ln(1/\sqrt{3})\cdot\frac{\pi}{6} - \underbrace{0} \quad \text{from (i)} \quad ✓$$

$$= \frac{\pi\ln\sqrt{3}}{3} + \frac{\pi\ln\sqrt{3}}{6}$$

$$= \frac{\pi\ln\sqrt{3}}{2}$$

$$= \frac{\pi\ln 3}{4} \quad ✓$$

Question 13

(a) $L_1: \underset{\sim}{r} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and $L_2: \underset{\sim}{r} = \begin{pmatrix}-1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\1\\-1\end{pmatrix}$

$$\begin{pmatrix}2\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} = 2\times1 + -1\times1 + 1\times-1 = 0$$

$\therefore L_1 \perp L_2$ ✓

Then, POI when $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\1\\-1\end{pmatrix}$

$$3+2\lambda = -1+\mu \quad — (1)$$
$$2-\lambda = 1+\mu \quad — (2)$$
$$-1+\lambda = -\mu \quad — (3)$$

(1)-(2): $1+3\lambda = -2$

$\lambda = -1 \quad — (4)$

Sub (4) in (2): $2--1 = 1+\mu$

$\mu = 2$ ✓

Test $\lambda=-1$, $\mu=2$ in (3): LHS $= -1-1 = -2$

RHS $= -2$

$\therefore$ POI at $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix} - 1\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$= \begin{pmatrix}1\\3\\-2\end{pmatrix}$ ✓

(b)(i) $F = ma$

$$-\frac{1}{10}\cancel{m}\sqrt{1+v} = \cancel{m}\ddot{x}$$

$$\ddot{x} = \frac{dv}{dt} = -\frac{1}{10}\sqrt{1+v}$$

$$\frac{dt}{dv} = \frac{-10}{\sqrt{1+v}}$$ ✓

$$\int_0^T dt = \int_{15}^{0} -10(1+v)^{-1/2}\,dv$$

$$[t]_0^T = 10\left[\frac{(1+v)^{1/2}}{1/2}\right]_0^{15}$$

$$T - 0 = 20(\sqrt{16} - \sqrt{1})$$

$\therefore T = 60$ seconds ✓

(ii) $$\ddot{x} = v\frac{dv}{dx} = -\frac{1}{10}\sqrt{1+v}$$

$$\frac{dv}{dx} = \frac{-\sqrt{1+v}}{10v}$$

$$\frac{dx}{dv} = \frac{-10v}{\sqrt{1+v}}$$ ✓

$$\int_0^D dx = \int_{15}^{0} \frac{-10v}{\sqrt{1+v}}\,dv$$

$$[x]_0^D = 10\int_0^{15} \frac{1+v-1}{\sqrt{1+v}}\,dv$$

$$D - 0 = 10\int_0^{15} (1+v)^{1/2} - (1+v)^{-1/2}\,dv$$ ✓

$$D = \left[\frac{(1+v)^{3/2}}{3/2} - \frac{(1+v)^{1/2}}{1/2}\right]_0^{15}$$

$$= \frac{2}{3}(16\sqrt{16} - 1\sqrt{1}) - 2(\sqrt{16} - \sqrt{1})$$

$\therefore D = 360$ m ✓

(c) (i) $$\frac{5}{(x+3)(2x+1)} = \frac{A}{x+3} + \frac{B}{2x+1}$$

$$5 = A(2x+1) + B(x+3)$$

Sub $x = -1/2$

$$5 = A(0) + B(5/2)$$

$$B = 2$$ ✓

Sub $x = -3$

$$5 = A(-5) + B(0)$$

$$A = -1$$ ✓

$\therefore A = -1,\ B = 2$

(ii) $$\int_0^2 \frac{5}{(x+3)(2x+1)}\,dx = \int_0^2 \frac{-1}{x+3} + \frac{2}{2x+1}\,dx$$

$$= \Big[-\ln|x+3| + \ln|2x+1|\Big]_0^2$$ ✓

$$= \left[\ln\left|\frac{2x+1}{x+3}\right|\right]_0^2$$

$$= \ln\left(\frac{5}{5}\right) - \ln\left(\frac{1}{3}\right)$$

$$= 0 - \ln(1/3)$$

$$= -\ln\left(\frac{1}{3}\right)$$

$$= \ln 3$$ ✓

(d) $S: \left| \underset{\sim}{r} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right| = 3$ and $L: \underset{\sim}{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

POI when

$$\left| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right| = 3$$

$$\left| \begin{pmatrix} 1+\lambda \\ 1+\lambda \\ -1+2\lambda \end{pmatrix} \right| = 3 \qquad |\underset{\sim}{a}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a} \qquad ✓$$

$$\begin{pmatrix} 1+\lambda \\ 1+\lambda \\ -1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1+\lambda \\ 1+\lambda \\ -1+2\lambda \end{pmatrix} = 9$$

$$(1+\lambda)^2 + (1+\lambda)^2 + (-1+2\lambda)^2 = 9$$

$$1+2\lambda+\lambda^2 + 1+2\lambda+\lambda^2 + 1-4\lambda+4\lambda^2 = 9$$

$$6\lambda^2 = 6$$

$$\lambda = \pm 1 \qquad ✓$$

Sub $\lambda = 1$ in L: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Sub $\lambda = -1$ in L: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

✓ Both POIs

Question 14

(a) (i) $x = A\sin(nt + \alpha)$

$A = 6$

$T = \frac{2\pi}{n} = 16$

$n = \frac{\pi}{8}$ ✓

At $t = 2$, $x = 6$

$6 = 6\sin\left(\frac{\pi}{4} + \alpha\right)$

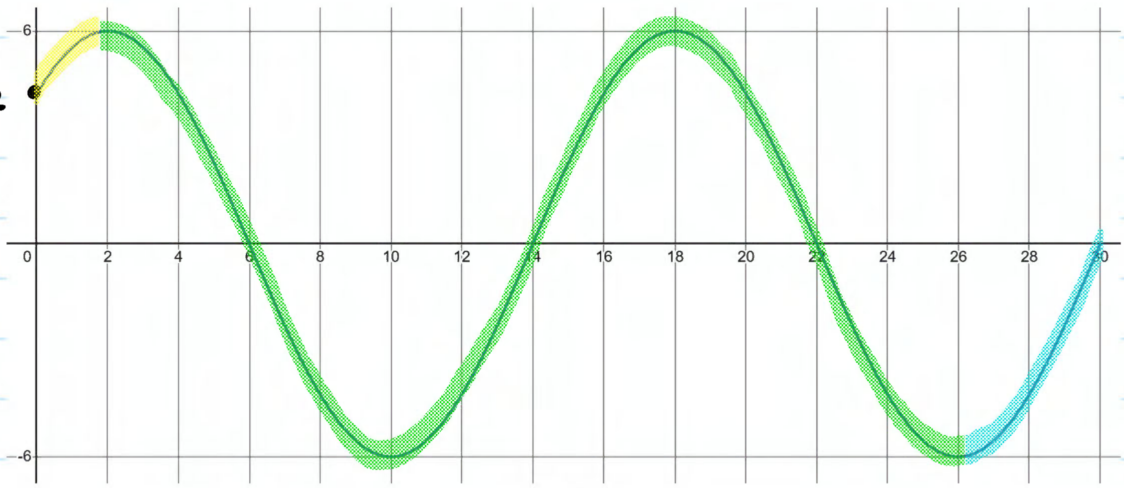
$\frac{\pi}{4} + \alpha = \sin^{-1}(1) = \frac{\pi}{2}$

$\alpha = \frac{\pi}{4}$

$\therefore x = 6\sin\left(\frac{\pi t}{8} + \frac{\pi}{4}\right)$ ✓

(ii)

$3\sqrt{2}$



At $t = 0$, $x = 6\sin\left(0 + \frac{\pi}{4}\right) = \frac{6}{\sqrt{2}} = 3\sqrt{2}$ ✓

Distance $= (6 - 3\sqrt{2}) + 3 \times 12 + 6$

$= 48 - 3\sqrt{2}$ metres ✓

(b) (i) RTP: $\tan 4\theta = \dfrac{4t(1-t^2)}{1-6t^2+t^4}$

LHS $= \tan 4\theta$

$= \tan(2\times 2\theta)$

$= \dfrac{2\tan 2\theta}{1-\tan^2 2\theta}$

$= \dfrac{2\left(2\tan\theta / 1-\tan^2\theta\right)}{1-\left(2\tan\theta / 1-\tan^2\theta\right)^2}$ Sub $t=\tan\theta$ ✓

$= \dfrac{4t / 1-t^2}{1 - 4t^2/(1-t^2)^2}$

$= \dfrac{\dfrac{4t}{\cancel{1-t^2}}}{\dfrac{1-2t^2+t^4-4t^2}{(1-t^2)^{\cancel{2}}}}$

$= \dfrac{4t(1-t^2)}{1-6t^2+t^4}$ ✓

$=$ RHS

(ii) $\tan\theta\tan 4\theta = 1$

$t \times \dfrac{4t(1-t^2)}{1-6t^2+t^4} = 1$ ✓

$4t^2(1-t^2) = 1-6t^2+t^4$

$4t^2 - 4t^4 = 1-6t^2+t^4$ ✓

$\therefore 5t^4 - 10t^2 + 1 = 0$

(iii) $t^2 = \dfrac{10\pm\sqrt{80}}{2(5)} = \dfrac{5\pm 2\sqrt{5}}{5}$

$t = \pm\sqrt{\dfrac{5\pm 2\sqrt{5}}{5}}$ ✓

But $0 < \tan\frac{\pi}{10} < \tan\frac{3\pi}{10}$

$\therefore \tan\frac{\pi}{10} = \sqrt{\dfrac{5-2\sqrt{5}}{5}}$ ✓

(c)(i) RTP: $\underbrace{\tan^{-1}\left(\frac{1}{k-1}\right)}_{\alpha} - \underbrace{\tan^{-1}\left(\frac{1}{k+1}\right)}_{\beta} = \tan^{-1}\left(\frac{2}{k^2}\right)$ for $k \geq 2$

$$\begin{cases} \alpha = \tan^{-1}\left(\frac{1}{k-1}\right) \\ \tan\alpha = \frac{1}{k-1} \\ \beta = \tan^{-1}\left(\frac{1}{k+1}\right) \\ \tan\beta = \frac{1}{k+1} \end{cases}$$

Consider $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$$= \frac{\frac{1}{k-1} - \frac{1}{k+1}}{1+\left(\frac{1}{k-1}\right)\left(\frac{1}{k+1}\right)} \quad ✓$$

$$= \frac{\frac{k+1-k+1}{\cancel{(k-1)(k+1)}}}{\frac{k^2-1+1}{\cancel{(k-1)(k+1)}}}$$

$$= \frac{2}{k^2} \quad ✓$$

$$\alpha - \beta = \tan^{-1}(2/k^2)$$

$$\therefore \tan^{-1}\left(\frac{1}{k-1}\right) - \tan^{-1}\left(\frac{1}{k+1}\right) = \tan^{-1}\left(\frac{2}{k^2}\right)$$

(ii) $\displaystyle\sum_{k=1}^{n} \tan^{-1}\left(\frac{2}{k^2}\right)$

$$= \tan^{-1}2 + \sum_{k=2}^{n} \tan^{-1}\left(\frac{1}{k-1}\right) - \tan^{-1}\left(\frac{1}{k+1}\right) \quad ✓$$

$$= \tan^{-1}2 + \underbrace{\tan^{-1}(1) - \tan^{-1}\left(\tfrac{1}{3}\right)}_{k=2} + \underbrace{\tan^{-1}\left(\tfrac{1}{2}\right) - \tan^{-1}\left(\tfrac{1}{4}\right)}_{k=3} + \underbrace{\tan^{-1}\left(\tfrac{1}{3}\right) - \tan^{-1}\left(\tfrac{1}{5}\right)}_{k=4}$$

$$+ \underbrace{\tan^{-1}\left(\tfrac{1}{4}\right) - \tan^{-1}\left(\tfrac{1}{6}\right)}_{k=5} + \ldots + \underbrace{\tan^{-1}\left(\tfrac{1}{n-2}\right) - \tan^{-1}\left(\tfrac{1}{n}\right)}_{k=n-1} + \underbrace{\tan^{-1}\left(\tfrac{1}{n-1}\right) - \tan^{-1}\left(\tfrac{1}{n+1}\right)}_{k=n}$$

$$= \tan^{-1}2 + \pi/4 + \tan^{-1}(1/2) - \tan^{-1}(1/n) - \tan^{-1}(1/n+1) \quad ✓$$

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n} \tan^{-1}\left(\frac{2}{k^2}\right)\right] = \lim_{n\to\infty}\left(\frac{\pi}{4} + \tan^{-1}2 + \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{n}\right) - \tan^{-1}\left(\frac{1}{n+1}\right)\right)$$

$$= \frac{\pi}{4} + \tan^{-1}2 + \tan^{-1}\left(\frac{1}{2}\right) - \lim_{n\to\infty}\left(\tan^{-1}\left(\frac{1}{n}\right) + \tan^{-1}\left(\frac{1}{n+1}\right)\right)$$

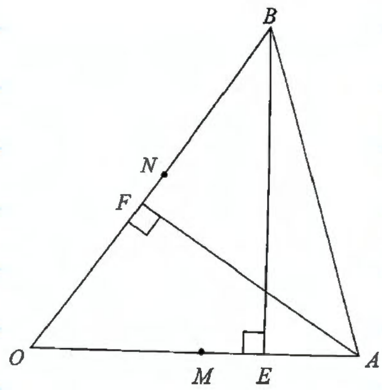
$$= \frac{\pi}{4} + \frac{\pi}{2} - 0$$

$\tan(A+B) = \dfrac{2+\frac{1}{2}}{1-(2)(\frac{1}{2})}$ = undefined ✓

$$= \frac{3\pi}{4}$$

OR... $\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$, $x>0$

Question 15



(a) $\vec{OM} = \frac{1}{2}\vec{OA} = \frac{1}{2}\underset{\sim}{a}$

$\vec{OE} = \lambda\vec{OA} = \lambda\underset{\sim}{a}$ λ scalar

$\vec{ON} = \frac{1}{2}\vec{OB} = \frac{1}{2}\underset{\sim}{b}$

$\vec{OF} = \mu\vec{OB} = \mu\underset{\sim}{b}$ μ scalar

Now, $|OM||OE| = \frac{\lambda}{2}|\underset{\sim}{a}|^2$

$|ON||OF| = \frac{\mu}{2}|\underset{\sim}{b}|^2$

Also, $AF \perp OB$, so $\vec{AF}\cdot\vec{OB} = 0$

$(\mu\underset{\sim}{b} - \underset{\sim}{a})\cdot\underset{\sim}{b} = 0$

$\mu\underset{\sim}{b}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$

$\underset{\sim}{a}\cdot\underset{\sim}{b} = \mu\underset{\sim}{b}\cdot\underset{\sim}{b} = \mu|\underset{\sim}{b}|^2$

Similarly, $BE \perp OA$, so $\vec{BE}\cdot\vec{OA} = 0$

$(\underset{\sim}{b} - \lambda\underset{\sim}{a})\cdot\underset{\sim}{a} = 0$

$\underset{\sim}{a}\cdot\underset{\sim}{b} - \lambda\underset{\sim}{a}\cdot\underset{\sim}{a} = 0$

$\underset{\sim}{a}\cdot\underset{\sim}{b} = \lambda\underset{\sim}{a}\cdot\underset{\sim}{a} = \lambda|\underset{\sim}{a}|^2$

Then $\underset{\sim}{a}\cdot\underset{\sim}{b} = \lambda|\underset{\sim}{a}|^2 = \mu|\underset{\sim}{b}|^2$

$\frac{\lambda}{2}|\underset{\sim}{a}|^2 = \frac{\mu}{2}|\underset{\sim}{b}|^2$

$\therefore |OM||OE| = |ON||OF|$

(b) $I_n = \int_0^1 \frac{x^n}{x^2+1}\,dx$

(i) LHS $= I_n + I_{n-2}$

$$= \int_0^1 \frac{x^n}{x^2+1}\,dx + \int_0^1 \frac{x^{n-2}}{x^2+1}\,dx$$

$$= \int_0^1 \frac{x^{n-2}\cancel{(x^2+1)}}{\cancel{x^2+1}}\,dx \quad ✓$$

$$= \left[\frac{x^{n-1}}{n-1}\right]_0^1$$

$$= \frac{1}{n-1} - 0$$

$$= \frac{1}{n-1} \quad ✓$$

$=$ RHS

(ii) $$\int_0^1 \frac{x^2(x-1)^2}{x^2+1}\,dx = \int_0^1 \frac{x^2(x^2-2x+1)}{x^2+1}\,dx$$

$$= \int_0^1 \frac{x^4}{x^2+1} - 2\frac{x^3}{x^2+1} + \frac{x^2}{x^2+1}\,dx \quad ✓$$

$$= I_4 - 2I_3 + I_2$$

$$= \frac{1}{3} - 2\left(\frac{1}{2} - I_1\right) \qquad \text{from (i)}$$

$$= \frac{1}{3} - 1 + \int_0^1 \frac{2x}{x^2+1}\,dx \quad ✓$$

$$= -\frac{2}{3} + \left[\ln(x^2+1)\right]_0^1$$

$$= -\frac{2}{3} + \ln 2 - \ln 1 \quad ✓$$

$$= \ln 2 - \frac{2}{3}$$

(c) $f_n(x) = e^x - \left(1 + \sum_{r=1}^{n} \frac{x^r}{r!}\right) \quad n = 1, 2, 3, \ldots$

(i) $f_n(0) = e^0 - \left(1 + \sum_{r=1}^{n} \frac{0^r}{r!}\right)$

$= 1 - 1 - 0$

$= 0$ ✓

$f_{n+1}(x) = e^x - \left(1 + \sum_{r=1}^{n+1} \frac{x^r}{r!}\right)$

$= e^x - 1 - \left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}\right)$

$f_{n+1}'(x) = e^x - 0 - \left(\frac{1}{1!} + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \ldots + \frac{nx^{n-1}}{n!} + \frac{(n+1)x^n}{(n+1)!}\right)$

$= e^x - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}\right)$ ✓

$= e^x - \left(1 + \sum_{r=1}^{n} \frac{x^r}{r!}\right)$

$= f_n(x)$

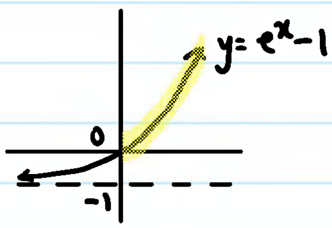
(ii) $f_1(x) = e^x - \left(1 + \frac{x^1}{1!}\right)$

$= e^x - 1 - x$

$f_1'(x) = e^x - 0 - 1$

$= e^x - 1$ ✓

> 0 for $x > 0$



$\therefore f_1(x)$ is increasing for $x > 0$

Since $f_1(0) = 0$, then $f_1(x) > 0$ for all $x > 0$

$e^x - 1 - x > 0$ ✓

$\therefore 1 + x < e^x$ for all $x > 0$

(iii) RTP: $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}<e^x$ for $x>0$.

Step 1 For $n=1$

LHS $=1+x$

RHS $=e$ $\quad$ [$1+x<e^x$ from (ii)]

LHS < RHS, statement is true for $n=1$ ✓

Step 2 Assume statement is true for $n=k$

i.e. $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^k}{k!}<e^x$

$\Rightarrow e^x-\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^k}{k!}\right)=f_k(x)>0$ ✓

Step 3 Attempt to prove statement is true for $n=k+1$

i.e. $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^{k+1}}{(k+1)!}<e^x$

$$\text{RHS}-\text{LHS}=e^x-\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^{k+1}}{(k+1)!}\right)$$

$=f_{k+1}(x)$

>0, since $f_{k+1}(0)=0$

and $f_{k+1}'(x)=\underline{f_k(x)>0}$ $\quad$ ✓ (i) and from assumption

$\therefore$ RHS > LHS

$\therefore$ Statement is true for $n=k+1$ if it is true for $n=k$

Step 4 By the principle of mathematical induction, the statement is true for integers $n\geq 1$.

Question 16

(a) (i) RTP: $(1+i\tan\theta)^n + (1-i\tan\theta)^n = \dfrac{2\cos n\theta}{\cos^n\theta}$ $\quad \cos\theta \neq 0,\ n \in \mathbb{Z}^+$

$$LHS = (1+i\tan\theta)^n + (1-i\tan\theta)^n$$
$$= \left(1+\frac{i\sin\theta}{\cos\theta}\right)^n + \left(1-\frac{i\sin\theta}{\cos\theta}\right)$$
$$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n \quad ✓$$
$$= \left(\frac{cis\theta}{\cos\theta}\right)^n + \left(\frac{cis(-\theta)}{\cos\theta}\right)^n$$
$$= \frac{cis(n\theta) + cis(-n\theta)}{\cos^n\theta}$$
$$= \frac{\cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)}{\cos^n\theta}$$

$\cos(-n\theta) = \cos(n\theta)$
$\sin(-n\theta) = -\sin(n\theta)$

$$= \frac{2\cos(n\theta)}{\cos^n\theta} \quad ✓$$
$$= RHS$$

(ii) Let $z = i\tan\theta$, $n=2$

$(1+z)^2 + (1-z)^2 = 0$ $\rightarrow$ Quadratic eqn, 2 roots

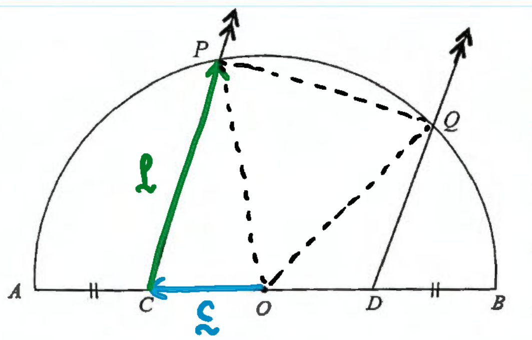
$$\frac{2\cos 2\theta}{\cos^2\theta} = 0$$
$$\cos 2\theta = 0$$
$$2\theta = \pi/2,\ 3\pi/2$$
$$\theta = \frac{\pi}{4},\ \frac{3\pi}{4} \quad ✓$$

$z_1 = i\tan\frac{\pi}{4}$

$z_2 = i\tan\frac{3\pi}{4} = i\tan\left(\frac{-\pi}{4}\right) = -i\tan\frac{\pi}{4}$ ✓

$\therefore z = \pm i\tan\frac{\pi}{4}$

(b)



(i) $\overrightarrow{DQ} = \lambda \overrightarrow{CP}$ $\quad (DQ \parallel CP)$ ✓ Explain

$\therefore \overrightarrow{DQ} = \lambda \underset{\sim}{p}$ for $\lambda > 0$ scalar

Now, $\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$

$= \underset{\sim}{c} + \underset{\sim}{p}$

$\overrightarrow{OQ} = \overrightarrow{OD} + \overrightarrow{DQ}$

$= -\underset{\sim}{c} + \lambda \underset{\sim}{p}$ ✓

Then, $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$ (equal radii of circle)

$(\underset{\sim}{c}+\underset{\sim}{p})\cdot(\underset{\sim}{c}+\underset{\sim}{p}) = (-\underset{\sim}{c}+\lambda\underset{\sim}{p})\cdot(-\underset{\sim}{c}+\lambda\underset{\sim}{p})$

$\cancel{\underset{\sim}{c}\cdot\underset{\sim}{c}} + 2\underset{\sim}{c}\cdot\underset{\sim}{p} + \underset{\sim}{p}\cdot\underset{\sim}{p} = \cancel{\underset{\sim}{c}\cdot\underset{\sim}{c}} - 2\lambda\underset{\sim}{c}\cdot\underset{\sim}{p} + \lambda^2\underset{\sim}{p}\cdot\underset{\sim}{p}$

$\underset{\sim}{p}\cdot\underset{\sim}{p}(1-\lambda^2) = -2\underset{\sim}{c}\cdot\underset{\sim}{p}(1+\lambda)$

$\underset{\sim}{p}\cdot\underset{\sim}{p}\cancel{(1+\lambda)}(1-\lambda) = -2\underset{\sim}{c}\cdot\underset{\sim}{p}\cancel{(1+\lambda)}$ ✓

$\therefore (1-\lambda)\underset{\sim}{p}\cdot\underset{\sim}{p} + 2\underset{\sim}{c}\cdot\underset{\sim}{p} = 0$

(ii) $\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{CD} + \overrightarrow{DQ}$

$= -\underset{\sim}{p} - 2\underset{\sim}{c} + \lambda\underset{\sim}{p}$

$= (\lambda-1)\underset{\sim}{p} - 2\underset{\sim}{c}$ ✓

Now, $\overrightarrow{CP}\cdot\overrightarrow{PQ} = \underset{\sim}{p}\cdot((\lambda-1)\underset{\sim}{p} - 2\underset{\sim}{c})$

$= (\lambda-1)\underset{\sim}{p}\cdot\underset{\sim}{p} - 2\underset{\sim}{c}\cdot\underset{\sim}{p}$

$= -[(1-\lambda)\underset{\sim}{p}\cdot\underset{\sim}{p} + 2\underset{\sim}{c}\cdot\underset{\sim}{p}]$

$= 0$ from (i) ✓

$\therefore CP \perp PQ$

$\therefore \angle CPQ = 90^\circ$

(c)(i) $I_n = \int_1^e (1-\ln x)^n \, dx$

$u = (1-\ln x)^n \qquad v' = 1$

$u' = \dfrac{n(1-\ln x)^{n-1}}{-x} \qquad v = x$

$= \left[x(1-\ln x)^n\right]_1^e - \int_1^e \dfrac{n(1-\ln x)^{n-1}}{-\cancel{x}} \cdot \cancel{x} \, dx$ ✓

$= e(1-\ln e)^n - 1(1-\ln 1)^n + n\int_1^e (1-\ln x)^{n-1} dx$

$= 0 - 1 + nI_{n-1}$ ✓

$= -1 + nI_{n-1}$

(ii) RTP: $I_n = n!e - 1 - \sum_{r=1}^{n} {}^nP_r \quad n \geq 1$

Step 1 For $n=1$

$LHS = I_1$

$= -1 + 1I_0$

$= -1 + \int_1^e dx$

$= -1 + e - 1$

$= e - 2$

$RHS = 1!e - 1 - {}^1P_1$

$= e - 1 - 1$

$= e - 2$

$\therefore LHS = RHS$, Statement is true for $n=1$ ✓

Step 2 Assume statement is true for $n=k$

i.e. $I_k = k!e - 1 - \sum_{r=1}^{k} {}^kP_r$

Step 3 Attempt to prove true for $n=k+1$

i.e. $I_{k+1} = (k+1)!e - 1 - \sum_{r=1}^{k+1} {}^{k+1}P_r$

$LHS = I_{k+1}$

$= -1 + (k+1)I_k$

$= -1 + (k+1)\left[k!e - 1 - \sum_{r=1}^{k} {}^kP_r\right]$ ✓

$= -1 + (k+1)!e - (k+1) - (k+1)\left[{}^kP_1 + {}^kP_2 + {}^kP_3 + \dots + {}^kP_k\right]$

PTO →

$$= -1 + (k+1)!e - (k+1) - (k+1)\left[\frac{k!}{(k-1)!} + \frac{k!}{(k-2)!} + \frac{k!}{(k-3)!} + \dots + \frac{k!}{(k-k)!}\right]$$

$$= -1 + (k+1)!e - (k+1) - \left(\frac{(k+1)!}{(k+1-2)!} + \frac{(k+1)!}{(k+1-3)!} + \frac{(k+1)!}{(k+1-4)!} + \dots + \frac{(k+1)!}{(k+1-(k+1))!}\right)$$

$$= -1 + (k+1)!e - (k+1) - \sum_{r=2}^{k+1} {}^{k+1}P_r$$

$$= -1 + (k+1)!e - (k+1) - \left(\sum_{r=1}^{k+1} {}^{k+1}P_r - {}^{k+1}P_1\right)$$

$$= -1 + (k+1)! - \cancel{(k+1)} - \sum_{r=1}^{k+1} {}^{k+1}P_r + \cancel{(k+1)}$$

$$= (k+1)!e - 1 - \sum_{r=1}^{k+1} {}^{k+1}P_r$$

$= RHS$

$\therefore$ Statement is true for $n = k+1$ if it is true for $n = k$

Step 4 By the principle of mathematical induction, statement is true for integers $n \geq 1$.